18.152 PROBLEM SET 1 SOLUTIONS

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1. Problem 4

Most students figured this problem out. Here is a general tip: whenever you cite a theorem, try to verify the conditions of that theorem carefully. In this way, you can realize why some assumptions in the problems are necessary.

Proof. Problem 4(1). The idea is to apply the maximum principle, either Theorem 1 or Corollary 2 from Lecture Note 1. As u is a smooth solution to the Cauchy-Neumann problem:

$$u_t(x,t) = u_{xx}(x,t),\ \forall x \in [0,L], t \geqslant 0, \tag{1}$$

$$u_x(x,t) = 0,\ \forall x \in \{0,L\}, t \geqslant 0, \tag{2}$$

$$u(x,0) = g(x),\ \forall x \in [0,L]. \tag{3}$$

By Corollary 2, for any $x \in [0,L]$ and $t \geqslant 0$, we have

$$u(x,t) \leqslant \max_{x\in[0,L]} g(x) \leqslant \max_{x\in[0,L]} |g(x)|.$$

Note that we can apply Corollary 2 also to $-u$, as it solves the equation

$$\begin{aligned}
(-u)_x(x,t) &= (-u)_{tt}(x,t),\ \forall x \in [0,L], t \geqslant 0,\\
(-u)_x(x,t) &= 0,\ \forall x \in \{0,L\}, t \geqslant 0,\\
(-u)(x,0) &= (-g)(x),\ \forall x \in [0,L].
\end{aligned}$$

As a result,

$$-u(x,t) \leqslant \max_{x\in[0,L]} (-g)(x) \leqslant \max_{x\in[0,L]} |g(x)|.$$

In all, we have $|u(x,t)| \leqslant \max_{x\in[0,L]} |g(x)|$ for any $x \in [0,L]$ and $t \geqslant 0$.

Problem 4(2). Set

$$w(t,x) = \frac{t}{1+t}|u_x|^2 + \frac{1}{2}u^2.$$

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By direct computation, we have

$$\begin{aligned} w_t &= \frac{1}{(1+t)^2}|u_x|^2 + \frac{2t}{1+t}u_x u_{tx} + u_t u, \\ w_x &= \frac{2t}{1+t}(u_x u_{xx}) + u_x u, \\ w_{xx} &= \frac{2t}{1+t}(u_{xx}^2 + u_x u_{xxx}) + u_x^2 + u_{xx} u \\ &= \frac{2t}{1+t}(u_{xx}^2 + u_x u_{xt}) + u_x^2 + u_t u, \end{aligned}$$

so

$$w_t - w_{xx} = -\frac{2t}{1+t}|u_{xx}|^2 - (1 - \frac{1}{(1+t)^2})|u_x|^2 \leqslant 0.$$

Problem 4(3). By Problem 4(2), the function w satisfies the following conditions:

$$w_t(x,t) \leqslant w_{xx}(x,t), \ \forall x \in [0,L], t \geqslant 0, \tag{4}$$

$$w_x(x,t) = 0, \ \forall x \in \{0,L\}, t \geqslant 0, \tag{5}$$

$$w(x,0) = \frac{1}{2}g^2(x), \ \forall x \in [0,L]. \tag{6}$$

Only the boundary condition (5) requires some explanation. Indeed, by (2),

$$w_x(x,t) = \frac{2t}{1+t}u_x u_{xx} + u_x u = 0 \text{ if } x = 0 \text{ or } L.$$

Now apply Corollary 2, so

$$\begin{aligned} \max_{x\in[0,L]} \frac{1}{2}g(x)^2 = \max_{x\in[0,L]} w(x,0) &\geqslant \max_{x\in[0,L]} w(x,t) \\ &\geqslant \max_{x\in[0,L]} \frac{t}{1+t}|u_x|^2(x,t). \end{aligned}$$

In other words,

$$|u_x|^2(x,t) \leqslant \frac{1+t}{2t} \max_{x\in[0,L]} g(x)^2$$

for any $x \in [0,L]$ and $t \geqslant 0$. □